THE QUASI-CLASSICAL MODEL OF THE SPHERICAL CONFIGURATION IN GENERAL RELATIVITY

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We consider the quasi-classical model of the spin-free configuration on the basis of the self-gravitating spherical dust shell in General Relativity. For determination of the energy spectrum of the stationary states on the basis of quasi-classical quantization rules it is required to carry out some regularization of the system. It is realized by an embedding of the initial system in the extended system with rotation. Then, the stationary states of the spherical shells are S-states of the system with the intrinsic momentum. The quasi-classical treatment of a stability of the configuration is associated with the Langer modification of a square of the quantum mechanical intrinsic momentum. It gives value of critical bare mass of the shell determining threshold of stability. For the shell with the bare mass smaller or equal to the Planck's mass, the energy spectra of bound states are found. We obtain the expression for tunneling probability of the shell and construct the quasi-classical model of the pair creation and annihilation of the shells.

1 Introduction

Thin spherical dust shell is among the simplest popular models of collapsing gravitating systems in General Relativity. The construction of the canonical formalism for this model, as well as its quantization, was considered in the works [1]-[8]. It was carried out in different ways and ultimately gave physically various results. There is a number of problems here, main of which is dependence on the choice of the evolution parameter (internal, external, proper). The choice of time coordinate affects the choice of a particular quantization scheme, leading, in general, to quantum theories which are not unitarily equivalent.

At a classical level, the choice of the evolutionary parameter is the choice of the observer, for which this parameter is the proper time. If we choose resting (interior or exterior) or comoving observer, we obtain the different pictures. Essential physical meaning has a picture of a gravitational collapse from the point of view of a distant observer at rest. In quantum theory this point of view enables us to treat bound states in the terms of asymptotic quantities and to build the relevant scattering theory correctly. On the other hand, the primordial black holes in the theory of self-gravitating shells is convenient for consideration from resting at the centre observer's point of view. In most of the works the canonical formalism is associated with proper time of a shell. The used reduction of the system leads to rather complicated Lagrangians and Hamiltonians, that creates difficulties at quantization. In particular it leads to the theories with higher derivative or to finite difference equations.

In our opinion, the choice of the exterior or interior resting observer is most natural. For providing of necessary properties of invariance, the indication of canonical

transformations of an extended phase space translating the corresponding dynamic systems one in to another is sufficient.

One of the most natural approaches to the quantization of this model it is reduction of the problem to the usual S- wave Klein-Gordon equation in a Coulomb field.[7,8] However, the initial classical Hamiltonian used here, in fact, was postulated.

The simple and natural variational principle for a dust shell in General Relativity can be constructed by elimination of the gravitational degrees of freedom from approaching action containing the standard Einstein-Hilbert term.[9] The variational formula, following from here, leads to the required Lagrangians. They describe the shell from the point of view of the exterior or interior resting observers. The condition of isometric equivalence of such descriptions of the shell leads to the momentum and Hamiltonian constraints. The corresponding dynamic systems are canonically equivalent in the extended phase space.

In the considered model, there was a set of problems, which we would like to discuss here. The basic problem, is why there are the stationary quantum states for the spherical dust configuration in General Relativity, where only an attraction takes place and there is not stable ground nonsingular state? Why do the bound states exist at bare mass m of a shell no more than the Planck's mass only? Why is the usual quantum mechanical consideration inapplicable at masses more than the Planck's mass m_{pl}? How and why does the statement of boundary conditions near a singularity depend from the free parameter of the theory, the bare mass m?

Let us note, that uncertainty relation is used for qualitative treatment of this kind of problems. For example, the energy of a ground state of a quantum oscillator can simply be obtained by means of this relation.[10] However, for the systems under consideration having singular, non-quadratic potentials, the straightforward application of an uncertainty principle is difficult.

Another approach gives the quasi-classical consideration. If we introduce a rotating shell, it is possible to note similarity of the equations of the relativistic Kepler problem and rotating dust shell. However, if in the first one, L is the angular orbital momentum of a particle, so in the second, L is an intrinsic momentum or spin of a shell. The mentioned similarity allows to look on the spherical quantum collapse from the quasi-classical point of view, similarly to the Kepler problem.

At the same time, we must keep in mind that the expression for a square of the quantum-mechanical angular momentum $L_q^2 = \hbar^2 l(l+1),\ (l = 0, 1, 2, ..)$ should be replaced by the quasi-classical expression $L_{sc}^2 = \hbar^2(l+1/2)^2$ in all the quasi-classical computations.[10,11] In the problems of the considered type such quasi-classical replacement gives an exact asymptotic of wave function for the radial equation following from the Schrodinger or Klein-Gordon equation. This replacement is justified by Langer transformation.[12,13]

The mentioned Langer modification can be received from the radial Klein-Gordon equation for the relativistic dust rotating shell. The characteristic relation connecting bare and Schwarzschild masses (m and M), intrinsic momentum L, radial adiabatic invariant I_R and principal quantum number n (see Appendix A) follows from this equation. Hence, it follows that the contribution in the radial adiabatic invariant gives not the square of the quantum-mechanical intrinsic momen-

tum $L_q^2 = \hbar^2 l(l+1)$, but its "renormalized" quasi-classical value $L_{sc}^2 = L_q^2 + \hbar^2/4$ where $\hbar^2/4$ is the Langer correction. This characteristic relation can be used as a quasi-classical quantization rule. From the point of view of such approach, the Bohr-Sommerfeld rule of the orbit quantization for the hydrogen-like atom follows from the characteristic equation and is exact (without taking into consideration a spin of the electron). The rule of this kind, we shall be using to the shell as well. In addition, the quantum states of the radially moving shell are determined as S- state of the shell with an intrinsic momentum.

According to this treatment, for the determination of an energy spectrum of the relativistic self-gravitating spherical shell it is necessary to make a transition to the shell with rotation. Then, the quasi-classical S- states ($l = 0$) of such a system differ from the corresponding classical states of the radially moving spherical shell by that there is a "residual intrinsic momentum" $L_{sc0} = \hbar/2$, which gives the essential contribution to investigated processes. An effective centrifugal potential of a repulsion which be opposed to gravitational self-action of the shell corresponds to this momentum. Thus, in quasi-classical language, the mechanism of a stability of the self-gravitating configuration is formed owing to "renormalization" of the contribution of an intrinsic momentum in the radial adiabatic invariant. The quantity of a "residual intrinsic momentum " $L_{sc0} = \hbar/2$ assigns value of the critical bare mass of the shell $m_k = m_{pl}$, determining threshold of its stability. Notice that formal application of the quasi-classical quantization rules to the radially moving shell (similarly to a nonrelativistic case (see the Appendix C)) leads to the divergent integral. Therefore, the mentioned transition to the rotating configuration can be considered as procedure of a regularization of the system.

Within the framework of such a quasi-classical approach, in this work one considers the model of a configuration on the basis of the self-gravitating spherical dust shell. We find an energy spectrum of the bound states of such a system. We obtain the formula of tunneling probability of the shell with negative energy from one region of the space in another. One constructs the quasi-classical model of the pair creation and annihilation of the shells.

The paper is organized as follows. In Sec.II we consider the dust spherical shell in General Relativity and its Lagrange and Hamiltonian formulation. The embedding of the considered two-dimensional dynamic system into the extended four-dimensional dynamic system is constructing. For this purpose we introduce the system with proper rotation and, necessary for its description, the collective angular coordinates. We treat a new system as some approximation to the shell with proper rotation. Then we carried out qualitative analysis and classification of the trajectories of the constructed system. In Sec.III we find an energy spectrum of bound states of the extended system and shells. Then one considers the quasi-classical treatment of a stability of the shell at $m < m_{pl}$ and at $m = m_{pl}$ and the statement of the boundary conditions for the quantum mechanical equation of the shell. In Sec.IV the quasi-classical model of the shell tunneling from one region into another of the analytically extended Schwarzschild space-time is constructed. The formula for the tunneling probability of the shell is obtained. In Sec.V we construct the quasi-classical model of the pair creation and annihilation of the shells.

In the Appendix A the characteristic relation associating the radial adiabatic invariant I_R, the parameters and conserved quantities of the theory (m, M, L) with a principal quantum number n has been obtained. In the Appendix B the indeterminacy relation is used for the obtaining of the energy of a ground state of the shell with the critical mass $m = m_k$. In the Appendix C, for comparison with a relativistic shell, the nonrelativistic dust spherical shell with $m \ll m_{pl}$ (very light shell) and its Bohr quantization are briefly considered.

In this work we study relativistic and non-relativistic systems as well. In this connection, we shall keep all the dimensional constants. Here c is the speed of light, γ is the gravitational constant, $\chi = 8\pi\gamma/c^2$, $\hbar$ is the Planck's constant. The metric tensor $g_{\mu\nu}$ ($\mu, \nu = 0, 1, 2, 3$) has the signature $(+\ -\ -\ -)$.

2 Spherical dust shell in General Relativity

2.1 Radially moving spherical shell

Let us consider a thin dust spherical shell with a surface density of bare mass σ in spherically-symmetric space-time $V^{(4)}$. The world leaf of the shell forms a timelike hypersurface Σ, which divides $V^{(4)}$ into exterior and interior regions $D_{\pm}^{(4)}$. Using the curvature coordinates, we can choose general spatial spherical coordinates $\{r,\ \theta,\ \varphi\}$ and individual time coordinates $t_{\pm}$ for $D_{\pm}^{(4)}$. Then the hypersurface Σ is given by the equation $r = R_-(t_-)$ in the interior region, and by $r = R_+(t_+)$ in the exterior one. When choosing appropriate times $t_{\pm}$ it is possible to put $R_-(t_-) = R_+(t_+)$. Thus, any particle of the shell is described by one collective dynamic coordinate $R = R_{\pm}(t_{\pm})$ and by two fixed individual (Lagrange) angular coordinates θ and φ.

The gravitational fields in regions $D_{\pm}^{(4)}$ are set by the metrics

$$ {}^{(4)}ds_{\pm}^2 = {}^{(2)}ds_{\pm}^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2)\,, \tag{1} $$

where

$$ {}^{(2)}ds_{\pm}^2 = f_{\pm}c^2 dt_{\pm}^2 - f_{\pm}^{-1}dr^2\,, \quad f_{\pm} = 1 - \frac{2\gamma M_{\pm}}{c^2 r}\,, \tag{2} $$

and M_+ and M_- are Schwarzschild masses. We suppose, that $R > 2\gamma M_{\pm}/c^2$.

The effective action for a shell can be represented in the two forms,[9]

$$ I_{\pm} = \frac{1}{2}\int \mathcal{L}_{\pm} dt_{|\pm} = -\frac{1}{2}\int \big(mc\ {}^{(2)}ds \mp U_{(G)}dt\big)_{|\pm}\,, \tag{3} $$

where

$$ \mathcal{L}_{\pm} = -mc^2\sqrt{f_{\pm} - f_{\pm}^{-1}R_{t\pm}^2/c^2} \pm U_{(G)} \tag{4} $$

are the effective Lagrangians of the dust shell. The subscripts $(\pm)$ designate, that the marked quantities are calculated with respect to the exterior or interior coordinates, respectively. Further, $m = 4\pi\sigma R^2$ is a bare mass of the shell, $R_{t\pm} = dR/dt_{\pm}$ are velocities in coordinate frames $\{t_{\pm}, R\}$, and

$$ U_{(G)} = -\frac{\gamma m^2}{2R} \tag{5} $$

is effective potential energy of gravitational self-action of the shell.

For systems with Lagrangians (4), momenta and the Hamiltonians have the form

$$P_{\pm} = \frac{mR_{t\pm}}{f_{\pm}\sqrt{f_{\pm} - f_{\pm}^{-1}R_{t\pm}^2/c^2}} = \frac{m}{f_{\pm}}\frac{dR}{d\tau}\,, \tag{6}$$

$$H_{\pm} = c\sqrt{f_{\pm}(m^2c^2 + f_{\pm}P_{\pm}^2)} \mp U_{(G)} = mc^2 f_{\pm}\,\frac{dt_{\pm}}{d\tau} \mp U_{(G)}\,, \tag{7}$$

where τ is proper time of the shell.

The dynamic systems with Lagrangians $\mathcal{L}_{\pm}$ are not independent. They satisfy the momentum and Hamiltonian constraints [9]

$$f_+P_+ = f_-P_-\,, \tag{8}$$

$$H_+ = H_- = (M_+ - M_-)c^2\,. \tag{9}$$

In addition, these dynamic systems are canonically equivalent in the extended phase space of the variables $\{P_{\pm},\ H_{\pm},\ R,\ t_{\pm}\}$. It is pointing out that the Lagrangians $\mathcal{L}_{\pm}$ describe the same system, but from the different resting observer point of view. The transition from the exterior observer to the interior one and vice versa induces discrete gauge transformations

$$M_{\pm} \leftrightarrow M_{\mp} \quad (f_{\pm} \leftrightarrow f_{\mp}), \qquad U_{(G)} \leftrightarrow -\ U_{(G)}, \qquad t_{\pm} \leftrightarrow t_{\mp}\,. \tag{10}$$

For a self-gravitating shell $M_- = 0$. Having introduced a designation $M_+ = M$, we have $f_- = 1,\ f_+ = 1 - 2\gamma M/c^2R$. Then, constraints (8), (9) reads

$$P_- = \left(1 - \frac{2\gamma M}{c^2R}\right)P_+\,, \qquad H_- = H_+ = Mc^2\,. \tag{11}$$

In this case the interchange of the exterior on interior observers induces the transformations $M \leftrightarrow 0\ (f_+ \leftrightarrow 1),\ U_{(G)} \leftrightarrow -U_{(G)},\ t_+ \leftrightarrow t_-$. Using the canonical equivalence, we can choose that observer, from whose position the picture looks simpler. Here, we shall use the interior observer frame of reference, for which ${}^{(2)}ds_-^2 = {}^{(2)}ds_0^2 = c^2dt^2 - dR^2$. Therefore, for the action of the shell we have

$$I = \frac{1}{2}\int\limits_{\gamma}\mathcal{L}dt = -\frac{1}{2}\int\limits_{\gamma}\left(mc\ {}^{(2)}ds_0 + U_{(G)}dt\right). \tag{12}$$

In this case Lagrangian, Hamiltonian and momentum of the shell have the simplest and natural form

$$\mathcal{L}_- \equiv \mathcal{L} = -mc^2\sqrt{1 - \dot{R}^2/c^2} - U_{(G)}\,, \tag{13}$$

$$H_- \equiv H = c\sqrt{m^2c^2 + P_R^2} + U_{(G)}\,, \tag{14}$$

$$P_- \equiv P_R = \frac{m\dot{R}}{\sqrt{1 - \dot{R}^2/c^2}} = m\,\frac{dR}{d\tau}\,, \tag{15}$$

where we designated $t_- = t$, $\dot{R} = dR/dt$.

2.2 Spherical dust shell with rotation

Consider now extended variant of the initial model (12), we need in the consequent quasi-classical quantization. Assume, that the spherical dust shell, except for radial motions, can make also rotary motions around the centre, as a whole, not breaking spherical symmetry (i.e. as a rigid body) and, not perturbing the gravitational field. A new system can be considered, as some approximation to a relativistic dust shell with proper rotation. This approximation allows us, for the complete description of the system, to be restricted by the collective coordinates and, still, to consider the shell, as a dynamic system with a finite number of degrees of freedom. The introduced system, except for the collective radial coordinate $R = R(t)$, has still angular collective coordinates, which can be taken to be Euler angles $\{\theta = \theta(t),\ \psi = \psi(t),\ \varphi = \varphi(t)\}$.

For construction of the Lagrangian of the new system we shall take the expression for the kinetic energy of a rotary motion of a spherical top,[14]

$$T_{rot} = \frac{I}{2}\left(\dot{\theta}^2 + \dot{\psi}^2 + \dot{\varphi}^2 + 2\dot{\varphi}\dot{\psi}\cos\theta\right) . \tag{16}$$

Here I is a moment of inertia. Considering the shell, as a massive spherical hollow top with the surface density of mass $\sigma = \sigma(R, t)$ and radius $R = R(t)$, we find

$$I = 4\pi\sigma R^4 = mR^2 . \tag{17}$$

From (16) it can be seen, that the metric in configurational space of rotary degrees of freedom of the shell is defined by the differential quadratic form

$$dl^2 = R^2(d\theta^2 + d\psi^2 + d\varphi^2 + 2d\varphi d\psi\cos\theta) \tag{18}$$

which is the metric of the homogeneous Bianchi IX space.[15]

In addition, since the shell is still spherical, then the form of gravitational self-action of the shell $U_{(G)}$ is conserved. Therefore, the action of the new system can be obtained from the action (12) by the replacement

$$^{(2)}ds_0^2 \to\ ^{(5)}ds^2 = c^2dt^2 - dR^2 - R^2(d\theta^2 + d\psi^2 + d\varphi^2 + 2d\varphi d\psi\cos\theta) . \tag{19}$$

Then the Lagrangian of the extended "five-dimensional" system has the form

$$\mathcal{L} = -mc\sqrt{c^2 - \dot{R}^2 - R^2(\dot{\theta}^2 + \dot{\psi}^2 + \dot{\varphi}^2 + 2\dot{\varphi}\dot{\psi}\cos\theta)} - U_{(G)} , \tag{20}$$

The system being considered, in addition to energy, has the following integrals of motion

$$P_\psi = mcR^2\left(\frac{d\psi}{ds} + \cos\theta\,\frac{d\varphi}{ds}\right) = \text{const} , \tag{21}$$

$$P_\varphi = mcR^2\left(\frac{d\varphi}{ds} + \cos\theta\,\frac{d\psi}{ds}\right) = \text{const} , \tag{22}$$

$$P_\theta^2 + \frac{P_\psi^2 + P_\varphi^2 - 2P_\psi P_\varphi \cos\theta}{\sin^2\theta} = \text{const}\,. \tag{23}$$

Here $P_\psi,\ P_\varphi,\ P_\theta$ are momenta, which are conjugate to the coordinates $\theta,\ \psi,\ \varphi$. Using these conservation laws, we can reduce dimensionality of the system. Supposing in (21) $P_\psi = 0$, we have $d\psi = -\cos\theta d\varphi$. In this case the metric (19) reduces to the four-dimensional one

$$^{(5)}ds^2 \to\ ^{(4)}ds_0^2 = c^2dt^2 - dR^2 - R^2(d\theta^2 + \sin^2\theta d\varphi^2)\,. \tag{24}$$

The Lagrangian and Hamiltonian of the extended system, obtained herewith

$$\mathcal{L} = -mc\sqrt{c^2 - \dot{R}^2 - R^2(\dot{\theta}^2 + \sin^2\theta\ \dot{\varphi}^2)} - U_{(G)}\,, \tag{25}$$

$$H = c\sqrt{m^2c^2 + P_R^2 + P_\theta^2/R^2 + P_\varphi^2/R^2\sin^2\theta} + U_{(G)} = \text{const} \tag{26}$$

describe a particle of mass m, moving in the space-time of special relativity under the influence of the potential $U_{(G)}$. Here $P_R = mcdR/ds$, $P_\theta = mcR^2 d\theta/ds$, $P_\varphi = mcR^2\sin^2\theta d\varphi/ds$ are momenta, which are conjugate coordinates $R,\ \theta,\ \varphi$.

Using the expression $L^2 = P_\theta^2 + P_\varphi^2/\sin^2\theta = \text{const}$ for complete intrinsic momentum of the system, and (5), the Hamiltonian (26) can be rewritten in the form

$$H = c\sqrt{m^2c^2 + P_R^2 + \frac{L^2}{R^2}} - \frac{\gamma m^2}{2R}\,. \tag{27}$$

We suppose, that the Hamiltonian constraint (11) is conserved for the extended system as well. Therefore, using (27), it can be rewritten in the form

$$\left(Mc + \frac{\gamma m^2}{2cR}\right)^2 - P_R^2 - \frac{L^2}{R^2} - m^2c^2 = 0\,. \tag{28}$$

Hence we find

$$\left(\frac{dR}{cd\tau}\right)^2 = -V(R) \equiv \left(\frac{\gamma^2m^2}{4c^2} - \frac{L^2}{m^2}\right)\frac{1}{c^2R^2} + \frac{\gamma M}{c^2R} + \frac{M^2}{m^2} - 1\,, \tag{29}$$

where $V(R) \le 0$ is the effective potential. Zeros of this function, i.e. solutions to the equation

$$V(R_m) = 0\,, \tag{30}$$

determine the turning points R_m of the system. The formula for radial acceleration of the system

$$\frac{d^2R}{d\tau^2} = -\frac{c^2}{2}\frac{dV}{dR} = -\frac{\gamma M}{2R^2} - \left(\frac{\gamma^2m^2}{4c^2} - \frac{L^2}{m^2}\right)\frac{1}{R^3} \tag{31}$$

together with expression for velocity (29) allows to carry out the classification of the trajectories and to study their stability. Note, that the component of acceleration, proportional to $1/R^2$, is always directed to the centre. The sign of the component

of acceleration, proportional to $1/R^3$, depends on a relation between the potential energy $U_{(G)}$ of the gravitational attraction and relativistic "centrifugal energy" of repulsion

$$U_{(L)} = \frac{cL}{R}\,. \tag{32}$$

At sufficiently large distances from the centre the first term in (31) dominates and we have an attraction. At small distances a situation is different. If $\gamma m^2/2 > cL$, then

$$|U_{(G)}| > U_{(L)} \tag{33}$$

and the energy of a gravitational attraction is greater than energy of a centrifugal repulsion, that causes a falling on the centre, i.e. a gravitational collapse. When $|U_{(G)}| < U_{(L)}$, the repulsion predominates above an attraction and the falling on centre is impossible.

The connection between the Schwarzschild and bare masses in a turning point R_m has the form

$$M = -\frac{\gamma m^2}{2c^2 R_m} + \sqrt{m^2 + \frac{L^2}{c^2 R_m^2}}\,. \tag{34}$$

For a qualitative analysis of the shell dynamics it is convenient to introduce the variable

$$\mathcal{M} = \mathcal{M}(m,\ L,\ R) \equiv -\frac{\gamma m^2}{2c^2 R} + \sqrt{m^2 + \frac{L^2}{c^2 R^2}}\,. \tag{35}$$

Then the relation $V(R) \leq 0$ leads to a condition

$$M \geq \mathcal{M}(m,\ L,\ R)\,, \tag{36}$$

for the regions of classically admissible motions of the system with given $M,\ m,\ L$. The equality $M = \mathcal{M}$ is fulfilled in the turning points R_m.

The extended system is invariant under the scale transformation

$$(M,\ m,\ P,\ L,\ R,\ \tau) \longrightarrow (aM,\ am,\ aP\ ,a^2 L,\ aR,\ a\tau)\,. \tag{37}$$

Therefore, we can fix one of conserved parameters, for example L.

The character of the function $\mathcal{M}(R)$, at a given $L \neq 0$, is determined by value of bare mass m, that can easily be seen from an asymptotical behavior

$$\mathcal{M}(R) = \begin{cases} m - \gamma m^2/2c^2 R\,, & R \to \infty\,, \\ cm^2 R/2L + (cL - \gamma m^2/2)/c^2 R, & R \to 0\,. \end{cases} \tag{38}$$

Hence we see, that at $R \to 0$, the function $\mathcal{M} = \mathcal{M}(R)$ behaves as follows

$$\begin{cases} \mathcal{M} \to +\infty, \text{ if } cL > \gamma m^2/2\,, \\ \mathcal{M} \to 0, \quad\ \text{ if } cL = \gamma m^2/2\,, \\ \mathcal{M} \to -\infty, \text{ if } cL < \gamma m^2/2\,. \end{cases} \tag{39}$$

Therefore the function $\mathcal{M}$ has the three types of behaviour. In the coordinates $\{\mathcal{M},\ R\}$, these cases are illustrated in Fig. 1. Here, the asymptotes $\mathcal{M} = \pm m$ are

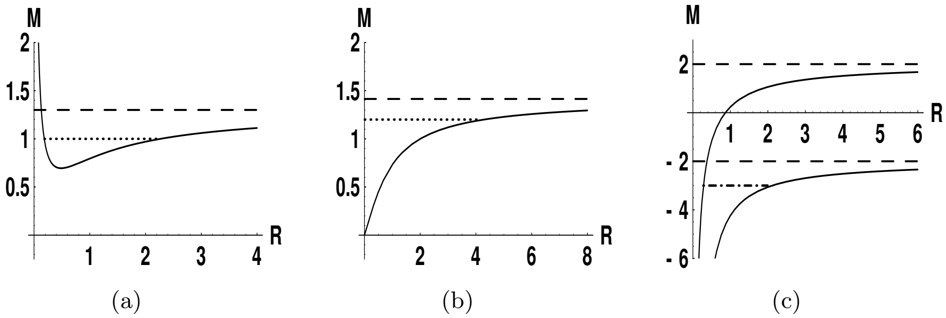

Figure 1: The plots of the function $\mathcal{M} = \mathcal{M}(R)$ for different types of the dust shells: (a) the light shell $0 < m < m_k$, (b) the shell with the critical mass $m = m_k$, (c) the massive shell $m > m_k$. Here we have put that $c = \gamma = L = 1$, therefore $m_k = \sqrt{2}$.

shown by dashed lines. The regions of the admissible motions are determined by segments of an axis R, for which the corresponding ordinates of points of the direct $\mathcal{M} = M$ lay above the ordinates of points of a curve $\mathcal{M} = \mathcal{M}(R)$ (for the lower curve in a Fig. 1(c) it is on the contrary). The turning radii R_m can be found as an abscissa of intersection points of a curve $\mathcal{M} = \mathcal{M}(R)$ and of a direct $\mathcal{M} = M$.

Introduce a critical bare mass of the shell corresponding to a momentum L,

$$m_k = \sqrt{\frac{2cL}{\gamma}}\,. \tag{40}$$

The relation between m and m_k determines the three possible basic types of the shells.

(i) Light shell: $0 < m < m_k$. Here $U_{(L)} > |U_{(G)|}$ and we have an infinite potential barrier in an origin of coordinates (Fig. 1(a)). According to the value of M we distinguish the variants:

(a) $M > m$ is a hyperbolic case. The shell with non-vanishing initial velocity falls from infinity, reaches the turning radius

$$R_1 = \frac{-\gamma M m^2 + \sqrt{\gamma^2 m^6 + 4c^2L^2(M^2 - m^2)}}{2c^2(M^2 - m^2)} \tag{41}$$

and returns. The requirement $c^2L^2 > \gamma^2 m^4/4$ guarantees, that $R_1 > 0$.

(b) $M = m$ is a parabolic case. The shell falls from the state of rest at infinity, reaches the turning radius

$$R_2 = \frac{1}{\gamma m}\left(\frac{L^2}{m^2} - \frac{\gamma^2 m^2}{4c^2}\right) \tag{42}$$

and again returns to infinity.

(c) $M_{min} \le M < m$ is an elliptic case. The shell goes inside of the "potential well" between turning radiuses

$$R_3 \le R \le R_4\,, \qquad R_{3,4} = \frac{\gamma M m^2 \pm \sqrt{\gamma^2 m^6 - 4c^2 L^2 (m^2 - M^2)}}{2c^2 (m^2 - M^2)}\,. \tag{43}$$

The requirement that the roots $R_{3,4}$ must be real together with the inequality $|U_{(G)}| < U_{(L)}$ leads to the following conditions

$$m_k^4 \left(1 - \frac{M^2}{m^2}\right) < m^4 < m_k^4\,. \tag{44}$$

The complete mass of the system M can not be less than the quantity

$$M_{min} = m \sqrt{1 - \frac{m^4}{m_k^4}}\,, \tag{45}$$

which is the extreme of the function $\mathcal{M}(R)$. Therefore, we have inequalities:

$$M_{min} < M < m < m_k\,. \tag{46}$$

(d) $M = M_{min}$ is a stationary case. The shell is at the bottom of the "potential wells". It has minimum mass M_{min} and the fixed radius

$$R_{ex} = \frac{2L^2}{\gamma m^4} M_{min} = \frac{2L^2}{\gamma m^3} \sqrt{1 - \frac{\gamma^2 m^4}{4c^2 L^2}}\,. \tag{47}$$

Thus, for a light shell $U_{(L)} > |U_{(G)}|$, $M \ge M_{min} > 0$, and thus, the gravitational collapse is impossible. If bare mass reaches a critical mass $m = m_k$, there occurs a bifurcation of the system, and the character of trajectories varies sharply.

(ii) Shell with a critical mass: $m = m_k$. Here $U_{(L)} = |U_{(G)}|$ and we have a finite potential well with the depth m (Fig. 2(b)). The acceleration (31) contains the component proportional to $1/R^2$ only. According to the value of M we distinguish the variants:

(a) $M > m$ is a hyperbolic case. The shell with non-vanishing initial velocity falls from infinity to centre.

(b) $M = m$ is a parabolic case. The shell falls on centre from the state of rest at infinity.

(c) $0 < M < m$ is an elliptic case. The system goes in a region

$$0 \le R \le R_5 = \frac{\gamma M m^2}{c^2 (m^2 - M^2)}\,. \tag{48}$$

(d)$M = 0$ is a "vacuum state". The system is characterized by the complete gravitational defect of mass, has the point sizes and rests at the centre.

Thus, mutual compensation of the centrifugal energy and of the gravitational energies of the self-action ($U_{(L)} = |U_{(G)}|$) is observed for the shell with a critical mass. It leads to the nonnegative complete mass of the shell and to a gravitational collapse. The further growth of bare mass leads to a new bifurcation of the system.

(iii) Massive shell: $m > m_k$. Here $|U_{(G)}| > U_{(L)}$ and we have an infinite potential well (Fig. 3(c)). Depending on the value of M, we have the following variants:

(a) $M > m$ is a hyperbolic case. The shell with nonvanishing initial velocity falls from infinity to the centre.

(b) $M = m$ is a parabolic case. The shell falls to the centre from a state of rest of infinity.

(c) $0 \leq M < m$ is an elliptic case. The system moves within the region

$$0 < R \leq R_6 = \frac{\gamma M m^2 + \sqrt{\gamma^2 m^6 - 4c^2 L^2 (m^2 - M^2)}}{2c^2 (m^2 - M^2)} \,. \tag{49}$$

The turning radius R_6 is real, if $m_k^4 (1 - M^2/m^2) < m^4$.

(d) $M = 0$ is a "vacuum state". The system is characterized by the complete gravitational defect of mass. Its motion with respect to the interior resting observer's clock, is described by the equation of an oscillator

$$\left(\frac{dR}{dt}\right)^2 + \omega^2 R^2 = \omega^2 R_7^2 \,, \tag{50}$$

where

$$\omega = \frac{2c^3}{\gamma m} \,, \quad R_7 = \sqrt{\frac{\gamma^2 m^2}{4c^4} - \frac{L^2}{m^2 c^2}} \,. \tag{51}$$

The shell falls from a point R_7 and in time $T = \pi \gamma m / 2c^3$ reaches the centre.

If one will prolong all the relations into the region of negative masses M, we should, for generality, take into account both signs before the radical in the expressions (34) and (35). It corresponds to the fact that the states with negative bare mass m are allowed. Then the following variants can be added:

(e) $-m < M < 0$ is a "state with a negative energy" and with region of motions

$$0 < R \leq R_8 = \frac{-\gamma |M| m^2 + \sqrt{\gamma^2 m^6 - 4c^2 L^2 (m^2 - M^2)}}{2c^2 (m^2 - M^2)} \,. \tag{52}$$

(f) $M \leq -|m|$ is a "state with a negative energy and with the two allowed regions of motions"

$$\{0 < R \leq R_9\} \,, \text{ if } m > m_k \,, \quad \{R_{10} \leq R < \infty\} \,, \text{ if } m < -m_k \,, \tag{53}$$

where

$$R_{9,10} = \frac{\gamma|M|m^2 \mp \sqrt{\gamma^2 m^6 + 4c^2 L^2 (M^2 - m^2)}}{2c^2(M^2 - m^2)} \,. \tag{54}$$

Thus, for the massive shell $|U_{(G)}| > U_{(L)}$, and, the gravitational collapse is inevitable.

The character of the motion of the system from the exterior observer point of view will be quite a different and depend on a cross arrangement of the turning radiuses and gravitational radius $R_g = 2\gamma M$. In the considered model the position of the interior observer is convenient for description of the primordial or eternal black holes, whereas the exterior observer position is useful for description of a gravitational collapse.

3 Energy spectrum of a bound states of the dust spherical shell

Consider now quasi-classical model of the like-particle spin-free configuration with mass M, using of a self-gravitating spherical dust shell with bare mass m. Classical action, Lagrangian, Hamiltonian, momenta and the constraints of this system are described by the formulas (11)-(15).

For determination of energy levels of stationary states of the system, the quasi-classical approach is used. Note, that formal application of the Bohr quantization rule of the elliptic orbits, in contrast to the nonrelativistic case (see Appendix C), leads to the divergent integral. Therefore, we shall perform some regularization of the system. For this purpose we shall take advantage of a method of embedding of the given two-dimensional dynamic system into the extended four-dimensional dynamic system. We suppose, that effective quantum dynamics of the radially moving shell with classical action (12) is an S- state of more general four-dimensional quantum system with the extended classical action, in which the angular degrees of freedom $\{\theta,\ \varphi\}$ are allowed also. The classical Lagrangian (25) and Hamiltonian (26) of such a system and the classification of the trajectories are considered in a Sec. II.2. Hence it follows that quasi-classical stationary states can have the light shell $m < m_k$, when $U_{(L)} > |U_{(G)}|$. The bound states are possible, when $M_{min} \leq M < m$, i.e. in the case of the elliptic trajectories 1.(c). The corresponding trajectory is shown on Fig. 1(a) by the dot line. Here the system is inside of the "potential wells" and goes into the restricted region.

We shall take an advantage of the following quantization rules,[10]

$$I_\varphi = \oint P_\varphi \, d\varphi = \int_0^{2\pi} P_\varphi \, d\varphi = 2\pi n_\varphi \hbar \,, \qquad (n_\varphi = 0, \ \pm 1, \ \pm 2, ...) \,, \tag{55}$$

$$I_\theta = \oint P_\theta \, d\theta = 2\int_{\theta_3}^{\theta_4} P_\theta \, d\theta = 2\pi\hbar \left(n_\theta + \frac{1}{2}\right) \,, \qquad (n_\theta = 0, \ 1, \ 2, ...) \,, \tag{56}$$

$$I_R = \oint P_R\, dR = 2\int\limits_{R_3}^{R_4} P_R\, dR = 2\pi\hbar\left(n + \frac{1}{2}\right)\,, \qquad (n = 0,\ 1,\ 2, ...)\,. \tag{57}$$

where $I_R,\ I_\theta,\ I_\varphi$ are adiabatic invariants, and $\{R_3,\ \theta_3\}$ and $\{R_4,\ \theta_4\}$ are the turning points coordinates (43) of the system. From (55) follows that $P_\varphi = n_\varphi\hbar$. Integral in the relation (56) gives $I_\theta = 2\pi(L - |P_\varphi|)$. Hence we obtain the quasi-classical expression for an intrinsic momentum

$$L = L_{sc} \equiv \hbar\left(l + \frac{1}{2}\right)\,, \qquad (l \equiv n_\theta + |n_\varphi| = 0, 1, 2, ..)\,. \tag{58}$$

Substituting P_R from (28) into (57), one obtains the radial adiabatic invariant

$$\begin{aligned} I_R &= 2\int\limits_{R_3}^{R_4} \sqrt{\left(cM + \frac{\gamma m^2}{2cR}\right)^2 - \frac{L^2}{R^2} - m^2c^2}\ dR \\ &= \frac{\pi}{c}\left(\frac{\gamma M m^2}{\sqrt{m^2 - M^2}} - \sqrt{4c^2L^2 - \gamma^2 m^4}\right)\,. \end{aligned} \tag{59}$$

In the Appendix A is shown, that the quantization rule (57), in which it should be put $L = L_{sc}$, is the consequence of the characteristic relation following from the Klein-Gordon equation (80) for stationary states of the extended system and is exact. Solving the relation (59) for M and taking into account (57) and (58), we obtain the energy spectrum of the extended system

$$\varepsilon_n = \mu\left\{1 + \mu^4\left[2n + 1 + \sqrt{(2l+1)^2 - \mu^4}\ \right]^{-2}\right\}^{-1/2}\,. \tag{60}$$

Here, using the Planck's units

$$m_{pl} = \sqrt{\frac{\hbar c}{\gamma}}\,, \qquad l_{pl} = \sqrt{\frac{\hbar\gamma}{c^3}}\,, \tag{61}$$

we come to dimensionless quantities

$$\varepsilon_n = \frac{M_n}{m_{pl}}\,, \qquad \mu = \frac{m}{m_{pl}}\,. \tag{62}$$

The required energy spectrum of bound states of the self-gravitating spherical dust shell, according to our rule, follows from (60) for S- states, i. e. at $l = 0$:

$$\varepsilon_n = \mu\left\{1 + \mu^4\left[2n + 1 + \sqrt{1 - \mu^4}\right]^{-2}\right\}^{-1/2}\,, \qquad (n = 0,\ 1,\ 2, ...)\,. \tag{63}$$

that coincides with the spectrum obtained by the Hamiltonian (14).[7]

The stability of the S-states of the shell with $m \leq m_{pl}$ can be connected with quantum indeterminacy of its position, which follows from the indeterminacy relation

$$\overline{(\triangle P_R)^2}\;\overline{(\triangle R)^2} \geq \frac{\hbar^2}{4}\,. \tag{64}$$

Here $\overline{(\triangle Y)^2} = \overline{(Y-\overline{Y})^2}$, where $\overline{Y}$ is a mean of the dynamic quantity $Y = \{P_R \text{ or } R\}$. Instead of estimate $\overline{(\triangle P_R)^2}$ and $\overline{(\triangle R)^2}$, we can find the manifestation of the quantum indeterminacy (64) having estimated its contribution in a "renormalization" of the adiabatic invariant I_R. The character of the contribution follows from the characteristic equation (86) (see Appendix A). We have two kinds of the contribution in a "renormalization" I_R. The first is the contribution to the value I_R, owing to $I_R \geq \pi\hbar$. The second is the contribution to argument I_R, which is equal $L_q^2 + \hbar^2/4$. It determines the effective "renormalized" value of a square of the intrinsic momentum $L_{sc}^2 = L_q^2 + \hbar^2/4$, therefore $L_{sc} \geq \hbar/2$. Owing to this we have effective "residual" intrinsic momentum $L_{sc0} = \hbar/2$ in the S-state.

Consideration of the "residual" intrinsic shell momentum can be treated quasi-classically as transition from a radially moving shell to the rotating shell, with the intrinsic momentum $L_{sc0} = \hbar/2$. The "residual" intrinsic momentum is appeared in the effective centrifugal energy (32) and in repulsion, corresponding to it. The appropriating effective critical mass, according to (61) and (40), turns out to be equal the Planck's mass $m_k = m_{pl}$. Therefore the condition $|U_{(G)}| < U_{(L)}$ takes the form

$$m < m_{pl} \quad \text{or} \quad \mu < 1\,. \tag{65}$$

Thus, in quasi-classical language, the mechanism of the stability of the self-gravitating configuration is formed owing to "renormalizations" of the contribution of an intrinsic momentum in the radial adiabatic invariant. The quantity $L_{sc0} = \hbar/2$ assigns value of the critical bare mass of the shell $m_k = m_{pl}$, determining threshold of its stability. For a light shell ($m < m_{pl}$) the effective centrifugal forces of the repulsion dominate above the gravitational self-action. The difference $U_{(L)} - |U_{(G)}|$ grows beyond all the bounds when approaching the centre. Therefore the infinite potential barrier, which prohibit a "falling on centre" even in S-state, is formed. For the shell with the critical bare mass ($m = m_{pl}$) we have $U_{(L)} = |U_{(G)}|$. From Fig. 1(b) it can be seen, that in this case there is a finite potential well. In the expression for the radial acceleration (31) there remains the term proportional $1/R^2$. In classical case it means a falling to the centre. This trajectory is shown on Fig. 1(b) by the dot line. In quantum case, owing to radial indeterminacy $I_R \geq \pi\hbar$ the stability of the system is conserved. The energy spectrum of stationary states of the shell follows from (63) at $\mu = 1$:

$$\varepsilon_n = \left\{1 + (2n+1)^{-2}\right\}^{-1/2}\,, \qquad (n = 0,\ 1,\ 2, ...)\,. \tag{66}$$

In classical theory the value $\varepsilon = 0$ corresponds to the ground state of such system, whereas in quantum theory the value $\varepsilon_0 = 1/\sqrt{2}$ does. Since the potential well is finite here, for an estimate of the energy of the ground state we can use the

indeterminacy relations (64) (see Appendix B). At last, for the massive shell ($m > m_{pl}$) we have $|U_{(G)}| > U_{(L)}$ and at the centre there is an infinite potential well. Therefore for any shell with $m > m_{pl}$ the "falling on the centre" takes place. The stationary states miss here, and the energy spectrum (63) lose its meaning, that is associated with losses of the bare mass of the shell similarly to quantum evaporation of black holes.[8]

Performing in (28) replacement $P_R \to -\imath\hbar d/dR$ and supposing $L = 0$, we obtain the radial Klein-Gordon equation for S-wave

$$\left[\frac{d^2}{dx^2} + \left(\varepsilon + \frac{\mu^2}{2x}\right)^2 - \mu^2\right]\psi = 0\,, \tag{67}$$

where we used the Planck's units (61) and variable $x = R/l_{pl}$. From our consideration follows, that the statement of the boundary conditions at the origin for this equation depends on the quantity of the bare mass m of the shell. Thus, the light shell ($m < m_{pl}$) has an infinite potential barrier at the origin (Fig. 1(a)). Therefore it is necessary to use boundary conditions $\psi(0) = 0$. The shell with the critical mass ($m = m_{pl}$) has a finite potential well (Fig. 1(b)), therefore it is sufficient to require a regularity $\psi(R)$ at the origin. In both cases the bound states are when $\varepsilon < \mu$.

For the massive shell ($m > m_{pl}$) it is impossible to be restricted to the one-particle approach already. Here probable tunnel transitions of the shell in other regions of the analytically extended space-time begin to play a role (see Sec.V).[8] In this case it is necessary the boundary conditions to formulate in the terms of the conserved Noether currents of the corresponding field theory.[7]

4 Tunneling dust spherical shell

In Sec. II.2 is noted, that under the conditions $m^2 > m_k^2$ (or $\gamma m^2/2 > cL$) and $M \leq -|m|$, there are two distinct trajectory of the shell corresponding to the different signs of the bare mass m (see case 3.(f) classifications). The shell can be either into interior or exterior non-overlapping regions (53) of the space (Fig. 1(c)), which are not cause relation. The transitions between these regions, forbidden in the classical theory, are admited in the quantum theory. These processes are possible only for the systems with binding energy $E_b = mc^2 - Mc^2 > 2mc^2$. Their quantum mechanical probability is estimated by the expression

$$W \sim \exp\left(-\Im\frac{I_R}{\hbar}\right) = \exp\left(-\frac{2}{\hbar}\,\Im\int\limits_{R_9}^{R_{10}} P_R\,dR\right)\,, \tag{68}$$

where R_9 and R_{10} are turning radii (54). The invariant I_R is calculated on of the classically forbidden trajectory, which corresponds to quantum mechanical tunneling through a potential barrier. This trajectory is shown on Fig. 1(c) by the dot-and-dash line.

Generally, the shell with negative Schwarzschild mass M is inconvenient for interpreting in the terms of the exterior removed observer.[7] However, the value $M \leq 0$ seems to be quite justified, if the shell is contained in the configuration from the set of the spherical shells inserted each into another.

Rewrite the above mentioned conditions as $|M| > |m| > m_k$, $M < 0$ and compare with conditions (46). Hence it can be seen, that we deal with Euclidean analog of the case of the stationary states considered in Sec. 3. The value I_R can be obtained from the expression (59) by an analytic continuation on parameters $\{M,\ m\}$ into region $|M| > |m| > m_k$, $M < 0$. It is achieved by replacement $M \to -|M|$. Supposing $L = \hbar/2$ and going to dimensionless unities (62), we have

$$I_R = \imath\pi\hbar \left(\frac{|\varepsilon|\mu^2}{\sqrt{\varepsilon^2 - \mu^2}} - \sqrt{\mu^4 - 1} \right) . \tag{69}$$

As a result, the tunneling probability is

$$W \sim \exp\left(\pi\sqrt{\mu^4 - 1} - \frac{\pi\mu^2}{\sqrt{1 - \mu^2/\varepsilon^2}} \right) \qquad (\varepsilon < 0) . \tag{70}$$

This formula gives an estimate of the tunneling probability of the dust spherical shell with negative energy from one region of the space in another. There are two thresholds at reaching which the process is possible here. The first threshold gives an energy level in the domain of the ground continuum $\varepsilon < -\mu < 0$. The second one concerns with the bare mass of the shell $|\mu| \geq 1$. From the formula (70) we see that at $|\varepsilon| \gg \mu$ the tunneling probability is determined by the bare mass only

$$W \sim \exp\left(\pi\sqrt{\mu^4 - 1} - \pi\mu^2 \right) . \tag{71}$$

In the case $\mu \gg 1$ the tunneling of the shell is, practically, a certain event, since $W \sim \exp(-\pi/2\mu^2) \to 1$.

5 The pair creation of the shells and its annihilation

Consider the manifestation of the above described tunnel transition in the quasi-classical model the pair creation of the spherical shells and their annihilation. It is constructed on the basis of self-gravitating system of two concentric dust shells.

Let $R_a,\ m_a,\ \tau_a$ are the radius, bare mass and proper time of the shell with the number a ($a = 1, 2$). Let $R_0 = 0,\ R_1 < R_2,\ R_3 = \infty$. We assume that M_a is the Schwarzschild mass defining gravitational field $f_a = 1 - 2\gamma M_a/c^2 r$ in the region $\{R_a < r < R_{a+1}\}$ ($a = 0, 1, 2$). We shall designate the coordinate time in this regions by t_a. Introduce the quantities $f_a^- = 1 - 2\gamma M_{a-1}/c^2 R_a$ and $f_a^+ = 1 - 2\gamma M_a/c^2 R_a$ ($a = 1, 2$). Then $P_a^\pm = m_a dR_a/f_a^\pm d\tau_a$ are the momenta of the shell with the number a, and $U_a^{(G)} = -\gamma m_a^2/2R_a$ is the energy of its gravitational self-action. The expression

$$H_a^\pm = c\varepsilon_a^\pm \sqrt{f_a^\pm (m_a^2 c^2 + f_a^\pm (P_a^\pm)^2)} \mp U_a^{(G)} \quad (a = 1,\ 2) , \tag{72}$$

where $\varepsilon_a^{\pm} = \pm 1$, determines the Hamiltonians of the a-th shell.[9] They are describing the dynamics of the a-th shell from the point of view of the exterior or interior resting observer in the regions $\{R_a < r < R_{a+1}\}$ or $\{R_{a-1} < r < R_a\}$, respectively. By virtue of the constraints

$$H_a^+ = H_a^- = (M_a - M_{a-1})\, c^2 \quad (a = 1,\ 2)\,, \tag{73}$$

the total Hamiltonian of the configuration satisfies to the relation

$$H = H_1^{\pm} + H_2^{\pm} = (M_2 - M_0)\, c^2\,. \tag{74}$$

The binding energy of the system is determined by the expression $E_b = (m_1 + m_2 + M_0 - M_2)\, c^2$.

For the self-gravitating configuration we have $M_0 = 0$. In addition, we assume that $M_2 = 0$, therefore $E_b = (m_1 + m_2)c^2$. Then the Hamiltonian constraints (73) read $H_1^{\pm} = M_1 c^2$ and $H_2^{\pm} = -M_1 c^2$, and the total Hamiltonian of the configuration vanishes $H = 0$. Further we are supposing that the Hamiltonian and bare mass of an exterior shell have positive values. Then $M_1 < 0$ and it is natural to introduce the designation $M_1 = -M$. From the Hamiltonians (72) and constraints (73) it follows, that $\varepsilon_1^+ = -1$, $\varepsilon_2^- = 1$. Taking into account, that $\varepsilon_a^{\pm} = 1$ corresponds to the increase of the radial coordinate in the direction of an exterior normal to the shell, it is easy to see, that $\varepsilon_2^+ = 1$. The sign ε_1^- remains uncertain, that corresponds to the different signs bare masses of the interior shell $\mu_1^- = \varepsilon_1^- m_1$.

Thus, the gravitational field of the configuration is described by the metric

$${}^{(4)}ds^2 = f c^2 dt^2 - f^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\alpha^2)\,, \tag{75}$$

where $f = 1 + 2\gamma M/c^2 r$, if $\{R_1 < r < R_2\}$, and $f = 1$, if $\{0 < r < R_1\}$ or $\{R_2 < r < \infty\}$.

Note, that each shell can be considered independently. In addition, taking into account canonical equivalence of description of the a-th shell with respect to the interior or exterior observers, it is possible to be restricted by the simplest from the pictures and, respectively, the simplest from the Hamiltonians $H_a^{\pm}$. Therefore we shall consider the Hamiltonians

$$H_1^- = \varepsilon_1^- m_1 c^2 \sqrt{1 + \left(\frac{dR_1}{c d\tau_1}\right)^2} - \frac{\gamma m_1^2}{2R_1} = -Mc^2\,, \tag{76}$$

$$H_2^+ = m_2 c^2 \sqrt{1 + \left(\frac{dR_2}{c d\tau_2}\right)^2} + \frac{\gamma m_2^2}{2R_2} = Mc^2\,, \tag{77}$$

that corresponds to the choice of the interior observer for the first shell and of the exterior observer for the second one. In the turning points $R_{m1},\ R_{m2}$ of the shells we have

$$M = -\varepsilon_1^- m_1 + \frac{\gamma m_1^2}{2R_{m1}c^2} = m_2 + \frac{\gamma m_2^2}{2R_{m2}c^2}\,. \tag{78}$$

Hence it follows, that if $\varepsilon_1^- = 1$, then $R_2 > R_1$ at $m_2 \geq m_1$. In the case of $\varepsilon_1^- = -1$, if $R_{m2} > R_{m1}$, then $m_2 > m_1$, and if $R_{m2} = R_{m1}$, then $m_2 = m_1 = m$. In the latter case the trajectories of the shells coincide. But their bare masses are equal in magnitude but opposite in sign, therefore the total bare mass vanishes, i.e. shells disappear.

In the previous section we considered the process, in which the shell with negative energy and in the state with $\mu_1^- = m$ as a result of the tunnel transition turned out in the state with $\mu_1^- = -m$. In accordance with that we shall consider the following situation. Let there is the self-gravitating system of two spherical dust shells with the equal bare mass $\mu_1^- = \mu_2^+ = m$ and with equal in magnitude but opposite in sign energies. The shells follow the trajectories $R_a = R_a(\tau_a)$ $(a = 1, 2)$ and their total energy is equal to zero $E = 0$. The interior shell, as a result of the tunnel transition, jumps at the trajectory of the exterior shell, but in the state with negative bare mass $\mu_1^- = -m$. Thus an annihilation of the shells take place. This process is possible at $m > m_{pl}$ and $M > m$ and its probability is described by the formula (70).

Naturally, the opposite event, i.e. the process of pair creation of shells, is possible as well. Imagine the following picture in the flat space-time: let two shells with equal in magnitude but opposite in sign bare masses and energies move, as a single object, along the coincident trajectories, which are determined by the Hamiltonians (76), (77). In the turning point this vacuum system breaks up into two distinct shells. The shell with positive bare mass and energy prolongs a motion, and the shell with negative bare mass and energies, as a result of the tunnel transition, jumps at the interior trajectory of the shell with negative energy, but with positive bare mass. As a result, of it there is the pair creation of the shells.

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Appendix A. Characteristic equation

To find connection between the radial adiabatic invariant I_R, quasi-classical and quantum mechanical intrinsic momenta, we shall consider the radial Klein-Gordon equation for the extended system. It can be received from the Hamiltonian constraints (28) by replacement

$$P \to \hat{P} = -\imath\hbar\frac{\partial}{\partial R}\,, \qquad L^2 \to L_q^2 = \hbar^2 l(l+1) \quad (l = 0, 1, 2, ...)\,. \tag{79}$$

We shall write out the equation for the radial wave function $\psi = \psi(R)$, using the Planck's units (61), dimensionless quantities (62) and variable $x = R/l_{pl}$:

$$\left[\frac{d^{\,2}}{dx^2} + \left(\varepsilon + \frac{\mu^2}{2x}\right)^2 - \frac{\Lambda_q^2}{x^2} - \mu^2\right]\psi = 0\,, \tag{80}$$

where $\Lambda_q^2 \equiv L_q^2/\hbar^2 = l(l+1)$.

The bound states are possible, when $\gamma m^2 \leq 2cL$ and $M < m$. We impose the boundary conditions $\psi(0) = 0$ and usual requirement of vanishing of a wave function ψ at infinity. As a result, the solution of the equation (80) is searched in the form

$$\psi = z^{\frac{1+S}{2}} \exp\left(-\frac{z}{2}\right) f(z)\,, \tag{81}$$

where $f(z)$ is some polynomial, and $z = 2x\sqrt{\mu^2 - \varepsilon^2}$. We choose s, so that

$$s = 2\sqrt{\Lambda_{sc}^2 - \frac{\mu^4}{4}}\,. \tag{82}$$

Here

$$\Lambda_{sc}^2 \equiv \frac{L_{sc}^2}{\hbar^2} = \Lambda_q^2 + \frac{1}{4}\,, \tag{83}$$

where, by definition, quantity

$$L_{sc}^2 \equiv \hbar^2 \Lambda_{sc}^2 = \hbar^2 \left(l + \frac{1}{2}\right)^2 \tag{84}$$

is the square of the quasi-classical intrinsic momentum. In this case, for $f(z)$ we obtain the equation in the canonical form

$$zf'' + (1 + s - z)f' + \left(\frac{I_R}{2\pi\hbar} - \frac{1}{2}\right) f = 0\,, \tag{85}$$

where

$$I_R = \pi\hbar \left(\frac{\varepsilon\mu^2}{\sqrt{\mu^2 - \varepsilon^2}} - \sqrt{4\Lambda_{sc}^2 - \mu^4}\right) \tag{86}$$

is the radial adiabatic invariant (59) rewritten in the Planck's units.

If the characteristic relation is fulfilled

$$\frac{I_R}{2\pi\hbar} - \frac{1}{2} = n \qquad (n = 0, 1, 2, ...), \tag{87}$$

then the solutions of the equation (85) are the Laguerre polynomials L_n^s.[16] Therefore

$$\psi = z^{\frac{1+S}{2}} e^{-\frac{z}{2}} L_n^s(z)\,, \tag{88}$$

and the energy spectrum (60) follows from the condition (87).

From (86) and (83) it can be seen, that the argument of the radial adiabatic invariant I_R contains not the square of the intrinsic momentum L_q^2, but the sum $L_{sc}^2 = L_q^2 + \hbar^2/4$. Thus, the quantum mechanical characteristic relation (87), associating both conserved quantity and quantum number n, is equivalent to the quasi-classical conditions of the quantization (57) and (56).

Appendix B. The shell with the critical mass and uncertainty relation

For a shell with the critical bare mass ($m = m_k$) there occurs almost complete compensation of a force of the gravitational self-action by the effective centrifugal force of the repulsion. Therefore the finite potential well of depth m_k at $R = 0$ is formed and any values of R are admited. Therefore, for an estimate of the ground state energy it is possible to use an uncertainty principle (64). In S-state we have $l = 0$, $L_{sc} = L_{sc0} = \hbar/2$, $m_k = m_{pl}$. Then the relation (27) taking into account (11) and (62) can be rewritten in the form

$$\varepsilon = \sqrt{1 + \frac{P_R^2}{c^2 m_{pl}^2} + \frac{1}{4x^2}} - \frac{1}{2x}\,. \tag{89}$$

Further, we suppose that $\overline{P_R} = 0$ and $\overline{R} = 0$. Therefore $\overline{(\triangle P_R)^2} = \overline{P_R^2}$, $\overline{(\triangle R)^2} = \overline{R^2}$ and we have

$$\overline{P_R^2} \geq \frac{\hbar^2}{4 l_{pl}^2 \overline{x^2}}\,. \tag{90}$$

Assuming, that (89) is satisfied, and for the mean-square quantities we have

$$\varepsilon \geq \sqrt{1 + \frac{1}{2z^2}} - \frac{1}{2z}\,, \tag{91}$$

where $z = \sqrt{(\overline{x^2})}$. The minimum of the energy $\varepsilon_0 = 1/\sqrt{2}$ is reached at $z = 1/\sqrt{2}$. This coincides with the energy of the critical shell (66) in the ground state at $n = 0$.

Appendix C. Very light spherical shell

For a very light shell the nonrelativistic approach is applicable. In this case the action for a thin dust shell moving in a proper gravitational field has the form:[9]

$$I_N = \int\limits_{t_1}^{t_2} \mathcal{L}_N \; dt = \int\limits_{t_1}^{t_2} \left(\frac{1}{2} m \dot{R}^2 + \frac{\gamma m^2}{2R} \right) \, dt\,. \tag{92}$$

Here $m = 4\pi\sigma R^2$, $R = R(t)$ is the mass and radius of the spherical shell, $\dot{R} = dR/dt$, $\sigma = \sigma(t)$. Hence, for momentum and Hamiltonian of a shell, we find

$$P = m\dot{R}\,, \qquad H = \frac{P^2}{2m} - \frac{\gamma m^2}{2R} = E = \text{const}\,. \tag{93}$$

With the help of the conservation law of energy, the requirement of applicability of nonrelativistic approach $\dot{R} \ll c$ can be written as a requirement to the size of the shell $R \gg R_c = \gamma m^2/(mc^2 - 2E)$. Here R_c is critical radius of a shell, at which its velocity reaches velocity of light c. For hyperbolic trajectories ($E > 0$) this radius can be somehow large and at $E = mc^2/2$ becomes infinite. For parabolic trajectories ($E = 0$) we have $R_c = \gamma m/c^2$. For elliptic trajectories ($E < 0$) the

system is localized inside the region $0 < R \leq R_{\mathrm{max}} = \gamma m^2/2|E|$, where R_{max} is a turning point of the system.

The critical values of parameters of a shell E, P, R ($R \sim R_{\mathrm{max}}$) and T ($T \sim$ of time of a collapse), at which the Newtonian approach becomes inapplicable, are equal to quantum nonrelativistic units of energy, momentum, length and time

$$E_\gamma = \frac{\gamma^2 m^5}{\hbar^2}\,, \qquad P_\gamma = \frac{\gamma m^3}{\hbar}\,, \qquad L_\gamma = \frac{\hbar^2}{\gamma m^3}\,, \qquad T_\gamma = \frac{\hbar^3}{\gamma^2 m^5} \tag{94}$$

for a particle of mass m in a gravitational field $\varphi = -\gamma m/R$. The nonrelativistic quantum approach is possible, if $|E| \sim E_\gamma \ll mc^2$. Hence it follows, that this is possible for a very light shell, when $m \ll m_{pl}$ ($R_{\mathrm{max}} \gg l_{pl}$).

In order to find an energy spectrum of the bound states of the nonrelativistic dust spherical shell with $E < 0$ it is sufficient to be restricted by the Bohr quantization rule of elliptic orbits. Since the shell moves radially, it is possible to consider its orbit as an limiting case of an elliptic orbit with two turning points. The first turning point is R_{max}, as a second point is $R = 0$. Indeed, the expression (92) can be treated as action for a particle of mass m, moving in a central gravitational field $\varphi = -\gamma m/R$. In the nonrelativistic case, at any non-vanishing orbital momentum, always, there is the second turning point R_{min} ($R_{\mathrm{min}} \leq R \leq R_{\mathrm{max}}$). In the considered approximation the turning point, which is the nearest to the centre and concerned with "residual" to the proper momentum $L_{sc0} = \hbar/2$, possibly to locate in the centre. Then the elliptic orbit of a particle is drawn out in a line segment connecting points $R = 0$ and R_{max}.

The application of the Bohr quantization rule to radial trajectories gives

$$\oint P\,dr = 2\int\limits_0^{R_{\mathrm{max}}} \sqrt{2m\left(E + \frac{\gamma m^2}{2R}\right)}\,dR = \frac{\pi\gamma m^{5/2}}{\sqrt{-2E}} = 2\pi n\hbar\,. \tag{95}$$

Hence, for the required spectrum of energies, we obtain

$$E_n = -\frac{E_\gamma}{8n^2} \qquad (n = 1, 2, 3, \ldots)\,. \tag{96}$$

For the very light shell ($m \ll m_{pl}$), by virtue of degeneration, similar to the nonrelativistic Kepler problem,[14] both of the above mentioned contributions of quantum indeterminacy are added (so, that $(n + 1/2) + 1/2 = n + 1$). Therefore the expression (63) for the energy spectrum transfers into expression (96), in which the reading of a quantum number n begins from unity rather than from zero. Here gravitational collapse is impossible, as the effective centrifugal energy always is greater than energy of gravitational self-action of the shell.

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